Lecture 28. Eigenvectors and linear transformations

Def Let $T: V \longrightarrow V$ be a linear transformation on a vector space V.

(1) A nonzero vector $\vec{v} \in V$ with $T(\vec{v}) = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$ is called an eigenvector of T with eigenvalue λ.

(2) An eigenbasis for T is a basis of V whose elements are all eigenvectors of T.

Note (1) If T is represented by a matrix A, the eigenvectors and eigenvalues of T are given by the eigenvectors and eigenvalues of A.

(2) We can use any basis of V to find the eigenvectors and eigenvalues of T.

Thm Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Given a basis $\mathcal{B}$ of $\mathbb{R}^n$, the standard matrix A and the $\mathcal{B}$-matrix $A_{\mathcal{B}}$ have the same characteristic polynomial.

pf Take B to be the matrix whose columns are given by $\mathcal{B}$.

$\Longrightarrow A = BA_{\mathcal{B}}B^{-1}$

$\Longrightarrow p_A(\lambda) = \det(A - \lambda I) = \det(BA_{\mathcal{B}}B^{-1} - \lambda I) = \det(BA_{\mathcal{B}}B^{-1} - \lambda BIB^{-1})$

$= \det(B(A_{\mathcal{B}} - \lambda I)B^{-1}) = \cancel{\det(B)} \det(A_{\mathcal{B}} - \lambda I) \cancel{\det(B^{-1})}$

$= p_{A_{\mathcal{B}}}(\lambda)$

$*$ $\det(B)\det(B^{-1}) = \det(BB^{-1}) = \det(I) = 1$

Ex Consider the linear transformation $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ given by

$$T(p(t)) = (t+1)\,p'(t) + p(t).$$

(1) Find all eigenvalues of T.

Sol The standard matrix has columns $[T(1)], [T(t)], [T(t^2)]$.

$p(t) = 1 : p'(t) = 0 \Rightarrow T(1) = (t+1)\cdot 0 + 1 = 1$

$p(t) = t : p'(t) = 1 \Rightarrow T(t) = (t+1)\cdot 1 + t = 1 + 2t$

$p(t) = t^2 : p'(t) = 2t \Rightarrow T(t^2) = (t+1)\cdot 2t + t^2 = 2t + 3t^2$

Hence the standard matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

A is triangular $\Rightarrow$ A has eigenvalues $\lambda = 1, 2, 3$ (diagonal entries)

$\Rightarrow$ T has eigenvalues $\boxed{\lambda = 1, 2, 3}$

(2) For each eigenvalue of A, find its algebraic multiplicity and geometric multiplicity.

Sol Each eigenvalue appears once on the diagonal.

$\Rightarrow$ Each eigenvalue has algebraic multiplicity 1

$\Rightarrow$ Each eigenvalue has geometric multiplicity 1

Hence the algebraic multiplicity and the geometric multiplicity are $\boxed{\text{both 1 for each eigenvalue}}$

(3) If possible, find an eigenbasis for T

Sol A is diagonalizable (3×3 matrix with 3 distinct eigenvalues)

$\Rightarrow$ There exists an eigenbasis for A.

$$A-I=\begin{bmatrix}0&1&0\\0&1&2\\0&0&2\end{bmatrix} \Rightarrow \mathrm{RREF}(A-I)=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$(A-I)\vec{x}=\vec{0} \Rightarrow \begin{cases}x_2=0\\x_3=0\end{cases} \underset{x_1=t}{\Rightarrow} \vec{x}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$A-2I=\begin{bmatrix}-1&1&0\\0&0&2\\0&0&1\end{bmatrix} \Rightarrow \mathrm{RREF}(A-2I)=\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$(A-2I)\vec{x}=\vec{0} \Rightarrow \begin{cases}x_1-x_2=0\\x_3=0\end{cases} \Rightarrow \begin{cases}x_1=x_2\\x_3=0\end{cases} \underset{x_2=t}{\Rightarrow} \vec{x}=t\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$A-3I=\begin{bmatrix}-2&1&0\\0&-1&2\\0&0&0\end{bmatrix} \Rightarrow \mathrm{RREF}(A-3I)=\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}$$

$$(A-3I)\vec{x}=\vec{0} \Rightarrow \begin{cases}x_1-x_3=0\\x_2-2x_3=0\end{cases} \Rightarrow \begin{cases}x_1=x_3\\x_2=2x_3\end{cases} \underset{x_3=t}{\Rightarrow} \vec{x}=t\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Hence an eigenbasis for A is given by

$$\vec{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}1\\1\\0\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$\Rightarrow$ An eigenbasis for T is given by

$$\boxed{p_1(t)=1,\ p_2(t)=1+t,\ p_3(t)=1+2t+t^2}$$

Ex Consider the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

respectively to $\vec{v} + \vec{w}$ and $-\vec{v} + 3\vec{w}$.

(1) Find all eigenvalues of T.

Sol Take $\mathcal{B}$ to be the basis of $\mathbb{R}^2$ given by $\vec{v}$ and $\vec{w}$.

$$\Rightarrow T(\vec{v}) = \vec{v} + \vec{w} = 1 \cdot \vec{v} + 1 \cdot \vec{w} \text{ and } T(\vec{w}) = -\vec{v} + 3\vec{w} = (-1) \cdot \vec{v} + 3 \cdot \vec{w}$$

$$\Rightarrow [T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } [T(\vec{w})]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The $\mathcal{B}$-matrix of T is

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

$$\Rightarrow p_{A_{\mathcal{B}}}(\lambda) = \lambda^2 - (1+3)\lambda + (1 \cdot 3 - (-1) \cdot 1) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$\Rightarrow A_{\mathcal{B}} \text{ has a unique eigenvalue } \lambda = 2$$

$$\Rightarrow T \text{ has a unique eigenvalue } \boxed{\lambda = 2}$$

(2) For each eigenvalue λ of A, find a basis of the λ-eigenspace

Sol For $\lambda = 2$, we consider $\mathrm{Nul}(A_{\mathcal{B}} - 2I)$.

$$A_{\mathcal{B}} - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathrm{RREF}(A_{\mathcal{B}} - 2I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A_{\mathcal{B}} - 2I)\vec{x} = \vec{0} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \xRightarrow[x_2 = t]{} \vec{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2-eigenspace in $\mathcal{B}$-coordinates

The 2-eigenspace has a basis given by

$$\vec{u} = -\vec{v} + \vec{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$$